

# THE ADDITIVE GROUP OF A LIE NILPOTENT ASSOCIATIVE RING

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**ABSTRACT.** Let  $\mathbb{Z}\langle X \rangle$  be the free unitary associative ring freely generated by an infinite countable set  $X = \{x_1, x_2, \dots\}$ . Define a left-normed commutator  $[x_1, x_2, \dots, x_n]$  by  $[a, b] = ab - ba$ ,  $[a, b, c] = [[a, b], c]$ . For  $n \geq 2$ , let  $T^{(n)}$  be the two-sided ideal in  $\mathbb{Z}\langle X \rangle$  generated by all commutators  $[a_1, a_2, \dots, a_n]$  ( $a_i \in \mathbb{Z}\langle X \rangle$ ). It can be easily seen that the additive group of the quotient ring  $\mathbb{Z}\langle X \rangle / T^{(2)}$  is a free abelian group. Recently Bhupatiraju, Etingof, Jordan, Kuszmaul and Li have noted that the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3)}$  is also free abelian. In the present note we show that this is not the case for  $\mathbb{Z}\langle X \rangle / T^{(4)}$ . More precisely, let  $T^{(3,2)}$  be the ideal in  $\mathbb{Z}\langle X \rangle$  generated by  $T^{(4)}$  together with all elements  $[a_1, a_2, a_3][a_4, a_5]$  ( $a_i \in \mathbb{Z}\langle X \rangle$ ). We prove that  $T^{(3,2)} / T^{(4)}$  is a non-trivial elementary abelian 3-group and the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$  is free abelian.

## 1. INTRODUCTION

Let  $\mathbb{Z}$  be the ring of integers and let  $\mathbb{Z}\langle X \rangle$  be the free unitary associative ring on the set  $X = \{x_i \mid i \in \mathbb{N}\}$ . Then  $\mathbb{Z}\langle X \rangle$  is the free  $\mathbb{Z}$ -module with a basis  $\{x_{i_1}x_{i_2}\dots x_{i_k} \mid k \geq 0, i_l \in \mathbb{N}\}$  formed by the non-commutative monomials in  $x_1, x_2, \dots$ . Define a left-normed commutator  $[x_1, x_2, \dots, x_n]$  by  $[a, b] = ab - ba$ ,  $[a, b, c] = [[a, b], c]$ . For  $n \geq 2$ , let  $T^{(n)}$  be the two-sided ideal in  $\mathbb{Z}\langle X \rangle$  generated by all commutators  $[a_1, a_2, \dots, a_n]$  ( $a_i \in \mathbb{Z}\langle X \rangle$ ).

It is clear that the quotient ring  $\mathbb{Z}\langle X \rangle / T^{(2)}$  is isomorphic to the ring  $\mathbb{Z}[X]$  of commutative polynomials in  $x_1, x_2, \dots$ . Hence, the additive group of  $\mathbb{Z}\langle X \rangle / T^{(2)}$  is a free abelian group. Recently Bhupatiraju, Etingof, Jordan, Kuszmaul and Li [2] have noted that the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3)}$  is also free abelian. In the present note we show that this is not the case for  $\mathbb{Z}\langle X \rangle / T^{(4)}$ .

Our first result is as follows.

**Theorem 1.1.** *Let  $v = [x_1, x_2, x_3][x_4, x_5]$ . Then  $3v \in T^{(4)}$  but  $v \notin T^{(4)}$ .*

Let  $T^{(3,2)}$  be the two-sided ideal of the ring  $\mathbb{Z}\langle X \rangle$  generated by all elements  $[a_1, a_2, a_3, a_4]$  and  $[a_1, a_2, a_3][a_4, a_5]$  where  $a_i \in \mathbb{Z}\langle X \rangle$ . Clearly,  $T^{(4)} \subset T^{(3,2)}$ .

Note that the ideal  $T^{(4)}$  is closed under all substitutions  $x_i \rightarrow a_i$  ( $i \in \mathbb{N}, a_i \in \mathbb{Z}\langle X \rangle$ ). By Theorem 1.1 we have  $3[a_2, a_3, a_4][a_5, a_6] \in T^{(4)}$ ; it follows that  $3a_1[a_2, a_3, a_4][a_5, a_6]a_7 \in T^{(4)}$  for all  $a_i \in \mathbb{Z}\langle X \rangle$ . Thus, we have

**Corollary 1.2.**  $T^{(3,2)}/T^{(4)}$  is a non-trivial elementary abelian 3-group.

Our second result is as follows.

**Theorem 1.3.** The additive group of the quotient ring  $\mathbb{Z}\langle X \rangle/T^{(3,2)}$  is free abelian.

Thus, the additive group of the ring  $\mathbb{Z}\langle X \rangle/T^{(4)}$  is a direct sum of a non-trivial elementary abelian 3-group  $T^{(3,2)}/T^{(4)}$  and a free abelian group isomorphic to  $\mathbb{Z}\langle X \rangle/T^{(3,2)}$ .

Let  $X_m = \{x_1, \dots, x_m\} \subset X$ ; then  $\mathbb{Z}\langle X_m \rangle \subset \mathbb{Z}\langle X \rangle$ . Let  $T_m^{(4)} = \mathbb{Z}\langle X_m \rangle \cap T^{(4)}$ ,  $T_m^{(3,2)} = \mathbb{Z}\langle X_m \rangle \cap T^{(3,2)}$ . By Theorem 1.1,  $T_m^{(4)} \subsetneq T_m^{(3,2)}$  if  $m \geq 5$ .

**Proposition 1.4.** If  $m = 2, 3, 4$  then  $T_m^{(4)} = T_m^{(3,2)}$ . In particular, for  $m \leq 4$  the additive group of  $\mathbb{Z}\langle X_m \rangle/T_m^{(4)}$  is free abelian.

Define  $\gamma_n = \gamma_n(\mathbb{Z}\langle X \rangle)$  by  $\gamma_1 = \mathbb{Z}\langle X \rangle$ ,  $\gamma_{n+1} = [\gamma_n, \mathbb{Z}\langle X \rangle]$  ( $n \geq 1$ ). Then  $\gamma_n$  is the  $n$ -th term of the lower central series of  $\mathbb{Z}\langle X \rangle$  viewed as a Lie ring. Clearly,  $T^{(n)}$  is the two-sided ideal of  $\mathbb{Z}\langle X \rangle$  generated by  $\gamma_n$ .

**Proposition 1.5.** Let  $w = [x_1[x_2, x_3, x_4], x_5]$ . Then  $w \in \gamma_3$ ,  $6w \in \gamma_4$  but  $w \notin \gamma_4$ .

Thus,  $w + \gamma_4$  is a non-trivial element of finite order (dividing 6) of the additive group of the quotient  $\gamma_3/\gamma_4$ .

Let  $f \in \mathbb{Z}\langle X \rangle$  be a multihomogeneous polynomial. It was conjectured in [2] that if  $f + \gamma_{l+1}$  is a torsion element in  $\gamma_l/\gamma_{l+1}$  then the degree of  $f$  is at least  $l + 3$  (Conjecture 5.3) and the degree of  $f$  with respect to each generator  $x_j$  is a multiple of the order of  $f + \gamma_{l+1}$  (Conjecture 5.2). Since  $w$  is of degree 5 and has degree 1 with respect to each  $x_i$ ,  $1 \leq i \leq 5$ , Proposition 1.5 gives a counter-example to these conjectures.

*Proof of Proposition 1.5.* By the identity (14) of [6], we have  $w \in \gamma_3$ . It can be deduced from the proof of [1, Lemma 6.1] that  $6w \in \gamma_4$ . On the other hand,  $w = x_1[x_2, x_3, x_4, x_5] + [x_1, x_5][x_2, x_3, x_4] \notin T^{(4)}$  because  $x_1[x_2, x_3, x_4, x_5] \in T^{(4)}$  and, by Theorem 1.1,  $[x_1, x_5][x_2, x_3, x_4] \notin T^{(4)}$ . Since  $\gamma_4 \subset T^{(4)}$ , we have  $w \notin \gamma_4$ , as required.  $\square$

**Remark 1.6.** The proof of Lemma 2.2 below shows that the reason behind the existence of 3-torsion in the additive group of  $\mathbb{Z}\langle X \rangle/T^{(4)}$  is the Jacobi identity. For this reason one might expect the structure of the additive group of  $\mathbb{Z}\langle X \rangle/T^{(n)}$  for arbitrary  $n > 4$  to be similar to that of  $\mathbb{Z}\langle X \rangle/T^{(4)}$ , that is, there should be an ideal  $I^{(n)} \subset \mathbb{Z}\langle X \rangle$  such that  $T^{(n)} \subset I^{(n)}$ , the quotient  $I^{(n)}/T^{(n)}$  being an elementary abelian 3-group (possibly trivial for some  $n$ ) and the additive group of  $\mathbb{Z}\langle X \rangle/I^{(n)}$  being free abelian. This might also suggest that the counter-example to Conjectures 5.2 and 5.3 of [2] given in Proposition 1.5 is in a certain sense exceptional, and that it may be possible to modify the conjectures slightly so that they would be true.

## 2. PROOF OF THEOREM 1.1

The following lemma is well-known (see, for instance, [5, Theorem 3.4], [7, Lemma 1], [8, Lemma 2]) but we prove it here in order to have the paper more self-contained.

**Lemma 2.1.** *For all  $a_1, \dots, a_5 \in \mathbb{Z}\langle X \rangle$ ,*

$$(1) \quad [a_1, a_2, a_3][a_4, a_5] + [a_1, a_2, a_4][a_3, a_5] \in T^{(4)},$$

$$[a_1, a_2, a_3][a_4, a_5] + [a_1, a_4, a_3][a_2, a_5] \in T^{(4)}.$$

*Proof.* Since  $[a, bc] = b[a, c] + [a, b]c$ ,  $[ab, c] = a[b, c] + [a, c]b$ , we have

$$\begin{aligned} [a_1, a_2, a_3 a_4, a_5] &= [a_3[a_1, a_2, a_4] + [a_1, a_2, a_3]a_4, a_5] \\ &= a_3[a_1, a_2, a_4, a_5] + [a_3, a_5][a_1, a_2, a_4] + [a_1, a_2, a_3][a_4, a_5] + [a_1, a_2, a_3, a_5]a_4. \end{aligned}$$

It is clear that  $[a_1, a_2, a_3 a_4, a_5]$ ,  $a_3[a_1, a_2, a_4, a_5]$ ,  $[a_1, a_2, a_3, a_5]a_4 \in T^{(4)}$  so  $[a_3, a_5][a_1, a_2, a_4] + [a_1, a_2, a_3][a_4, a_5] \in T^{(4)}$ . Further,  $[[a_1, a_2, a_4], [a_3, a_5]] = [a_1, a_2, a_4, a_3, a_5] - [a_1, a_2, a_4, a_5, a_3] \in T^{(4)}$  so

$$\begin{aligned} [a_1, a_2, a_3][a_4, a_5] + [a_1, a_2, a_4][a_3, a_5] \\ = [a_3, a_5][a_1, a_2, a_4] + [a_1, a_2, a_3][a_4, a_5] + [[a_1, a_2, a_4], [a_3, a_5]] \in T^{(4)}, \end{aligned}$$

as required.

Note also that

$$\begin{aligned} [a_5, a_2 a_4, a_1, a_3] &= [a_2[a_5, a_4] + [a_5, a_2]a_4, a_1, a_3] \\ &= [a_2[a_5, a_4, a_1] + [a_2, a_1][a_5, a_4] + [a_5, a_2][a_4, a_1] + [a_5, a_2, a_1]a_4, a_3] \\ &= a_2[a_5, a_4, a_1, a_3] + [a_2, a_3][a_5, a_4, a_1] + [a_2, a_1][a_5, a_4, a_3] + [a_2, a_1, a_3][a_5, a_4] \\ &\quad + [a_5, a_2][a_4, a_1, a_3] + [a_5, a_2, a_3][a_4, a_1] + [a_5, a_2, a_1][a_4, a_3] + [a_5, a_2, a_1, a_3]a_4. \end{aligned}$$

It is clear that  $[a_5, a_2 a_4, a_1, a_3]$ ,  $a_2[a_5, a_4, a_1, a_3]$ ,  $[a_5, a_2, a_1, a_3]a_4 \in T^{(4)}$ . Also, by (1),  $[a_2, a_3][a_5, a_4, a_1] + [a_2, a_1][a_5, a_4, a_3] \in T^{(4)}$  and  $[a_5, a_2, a_3][a_4, a_1] + [a_5, a_2, a_1][a_4, a_3] \in T^{(4)}$ . It follows that  $[a_2, a_1, a_3][a_5, a_4] + [a_5, a_2][a_4, a_1, a_3] \in T^{(4)}$ , therefore  $[a_1, a_2, a_3][a_4, a_5] + [a_1, a_4, a_3][a_2, a_5] \in T^{(4)}$ , as required.  $\square$

The following lemma is also well-known (see, for instance, [5], [7, Lemma 1], [9, Lemma 1]).

**Lemma 2.2.** *For all  $a_1, \dots, a_5 \in \mathbb{Z}\langle X \rangle$ , we have  $3[a_1, a_2, a_3][a_4, a_5] \in T^{(4)}$ .*

*Proof.* It is clear that if  $\sigma = (12)$  or  $\sigma = (45)$  then, for all  $a_1, \dots, a_5 \in \mathbb{Z}\langle X \rangle$ ,

$$[a_1, a_2, a_3][a_4, a_5] = -[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}][a_{\sigma(4)}, a_{\sigma(5)}].$$

On the other hand, if  $\sigma = (34)$  or  $\sigma = (24)$  then, by Lemma 2.1,

$$[a_1, a_2, a_3][a_4, a_5] \equiv -[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}][a_{\sigma(4)}, a_{\sigma(5)}] \pmod{T^{(4)}}.$$

Since the transpositions (12), (45), (34) and (24) generate the entire group  $S_5$  of the permutations of the set  $\{1, 2, 3, 4, 5\}$ , for all  $\sigma \in S_5$  we have

$$(2) \quad [a_1, a_2, a_3][a_4, a_5] \equiv \text{sgn}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}][a_{\sigma(4)}, a_{\sigma(5)}] \pmod{T^{(4)}}.$$

Now note that, by the Jacobi identity,

$$[a_1, a_2, a_3][a_4, a_5] + [a_2, a_3, a_1][a_4, a_5] + [a_3, a_1, a_2][a_4, a_5] = 0.$$

By (2), this equality implies  $3[a_1, a_2, a_3][a_4, a_5] \in T^{(4)}$  for all  $a_1, \dots, a_5 \in \mathbb{Z}\langle X \rangle$ , as required.  $\square$

**Remark 2.3.** It is known (see, for instance, [9, Lemma 1]) that

$$\begin{aligned} & [a_1, a_2, a_3][a_4, \dots, a_n, a_{n+1}] \\ & \equiv \text{sgn}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}][a_4, \dots, a_n, a_{\sigma(n+1)}] \pmod{T^{(n)}} \end{aligned}$$

for each  $n \geq 4$ , all  $a_i \in \mathbb{Z}\langle X \rangle$  and all permutations  $\sigma$  of the set  $\{1, 2, 3, (n+1)\}$ . The proof is similar to that of (2). It follows that

$$3[a_1, a_2, a_3][a_4, \dots, a_n, a_{n+1}] \in T^{(n)}$$

for all  $n \geq 4$  and all  $a_i \in \mathbb{Z}\langle X \rangle$ .

By Lemma 2.2, we have  $3v \in T^{(4)}$ . To prove Theorem 1.1 it remains to prove the following.

**Lemma 2.4.**  $v \notin T^{(4)}$ .

Recall that  $X_5 = \{x_1, x_2, x_3, x_4, x_5\} \subset X$ ,  $\mathbb{Z}\langle X_5 \rangle \subset \mathbb{Z}\langle X \rangle$ ,  $T_5^{(4)} = \mathbb{Z}\langle X_5 \rangle \cap T^{(4)}$ . Note that  $v = [x_1, x_2, x_3][x_4, x_5] \in \mathbb{Z}\langle X_5 \rangle$ . Let  $I$  be the ideal of  $\mathbb{Z}\langle X_5 \rangle$  spanned by all monomials  $x_{i_1}x_{i_2}\dots x_{i_k}$  ( $1 \leq i_1, i_2, \dots, i_k \leq 5$ ) such that  $i_r = i_s$  for some  $r \neq s$ . In particular, if  $k > 5$  then  $x_{i_1}x_{i_2}\dots x_{i_k} \in I$ . The following proposition will be proved in the next sections.

**Proposition 2.5.**  $v \notin T_5^{(4)} + I$ .

Lemma 2.4 is an immediate corollary of Proposition 2.5 and Theorem 1.1 follows immediately from Lemmas 2.2 and 2.4. This completes the proof of Theorem 1.1 provided that Proposition 2.5 is proved.

### 3. AUXILIARY RESULTS

Let  $P_n$  ( $n \leq 1$ ) be the subgroup of the additive group of  $\mathbb{Z}\langle X \rangle$  generated by all monomials which are of degree 1 in each variable  $x_1, \dots, x_n$  and do not contain any other variable. Then  $P_n$  is a free abelian group of rank  $n!$ . Let  $L(X)$  be the free Lie ring on the free generating set  $X$ ,  $L(X) \subset \mathbb{Z}\langle X \rangle$ . Define  $V_n = L(X) \cap P_n$ . The following lemma is well known (see, for instance, [3, Exercise 4.3.8]).

**Lemma 3.1.** *For each  $n > 1$ ,  $V_n$  is a free abelian group with a basis*

$$(3) \quad \left\{ [x_n, x_{i_1}, \dots, x_{i_{n-1}}] \mid \{i_1, \dots, i_{n-1}\} = \{1, 2, \dots, n-1\} \right\}.$$

*Proof.* It can be easily proved using the Jacobi identity that the commutators of (3) generate  $V_n$  as a subgroup of the additive group of  $\mathbb{Z}\langle X \rangle$ . On the other hand, the leading term (in the lexicographic order) of the commutator  $[x_n, x_{i_1}, \dots, x_{i_{n-1}}]$  is the monomial  $x_n x_{i_1} \dots x_{i_{n-1}}$ . Hence, distinct commutators of (3) have distinct leading terms. It follows that the elements of (3) are linearly independent so they form a basis of  $V_n$ .  $\square$

Let  $W_1$  be the subgroup of the additive group of  $\mathbb{Z}\langle X_5 \rangle$  generated by all elements  $x_{i_1}[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}]$  and let  $W_2$  the subgroup generated by the elements  $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}]$  and  $[x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}]$  where  $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$ . Note that

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}] = [x_{i_3}, x_{i_4}, x_{i_5}][x_{i_1}, x_{i_2}] + [[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}, x_{i_5}]]$$

so  $[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}] \in W_2$ .

**Lemma 3.2.**  $W_1 \cap W_2 = 0$ .

*Proof.* Let  $W_1^{(j)}$  ( $1 \leq j \leq 5$ ) be the subgroup of  $W_1$  generated by the elements  $x_j[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}]$  where  $\{j, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$ . It is clear that  $W_1 = W_1^{(1)} \oplus \dots \oplus W_1^{(5)}$ .

Let  $\nu_i$  be the endomorphism of  $\mathbb{Z}\langle X \rangle$  defined by  $\nu_i(x_i) = 1$ ,  $\nu_i(x_j) = x_j$  for all  $j \neq i$ . It is clear that  $\nu_i(W_2) = 0$  and  $\nu_i(W_1^{(j)}) = 0$  for all  $j \neq i$ . On the other hand,  $W_1^{(i)} \cap \text{Ker } \nu_i = 0$ . Indeed, suppose in order to simplify notation that  $i = 1$ . Then it follows easily from Lemma 3.1 that the set  $C_1 = \{x_1[x_5, x_{i_2}, x_{i_3}, x_{i_4}] \mid \{i_2, i_3, i_4\} = \{2, 3, 4\}\}$  is a basis of the free abelian group  $W_1^{(1)}$ . On the other hand, by the same lemma, the set  $\nu_1(C_1) = \{[x_5, x_{i_2}, x_{i_3}, x_{i_4}] \mid \{i_2, i_3, i_4\} = \{2, 3, 4\}\}$  is linearly independent. It follows that  $W_1^{(1)} \cap \text{Ker } \nu_1 = 0$ , as claimed. If  $i > 1$  then the proof is similar.

Now we are in a position to complete the proof of Lemma 3.2. Suppose that  $f \in W_1 \cap W_2$ . Since  $f \in W_2$ , we have  $f \in \text{Ker } \nu_i$  for all  $i$ . On the other hand,  $f \in W_1$  so  $f = f_1 + \dots + f_5$  where  $f_i \in W_1^{(i)}$ . For each  $i$  we have  $f_i \in \text{Ker } \nu_i$  because  $f \in \text{Ker } \nu_i$  and  $f_j \in \text{Ker } \nu_i$  for all  $j \neq i$ . Since  $W_1^{(i)} \cap \text{Ker } \nu_i = 0$ , we have  $f_i = 0$ . Thus,  $f = 0$  so  $W_1 \cap W_2 = 0$ . The proof of Lemma 3.2 is completed.  $\square$

Let

$$\mathcal{B}_1 = \{[x_5, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}\},$$

$$\mathcal{B}_2 = \{[x_{i_1}, x_{i_2}, x_{i_3}][x_5, x_{i_4}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, i_1 > i_2, i_3\},$$

$$\mathcal{B}_3 = \{[x_{i_1}, x_{i_2}][x_5, x_{i_3}, x_{i_4}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, i_1 > i_2\}$$

and let  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ .

**Lemma 3.3.**  $W_2$  is a free abelian group with a basis  $\mathcal{B}$ .

*Proof.* It can be easily seen that  $\mathcal{B} \subset W_2$  and  $\mathcal{B}$  generates  $W_2$  as a subgroup of the additive group of  $\mathbb{Z}\langle X_5 \rangle$ . On the other hand, the leading monomials (in the lexicographic order) of distinct elements of  $\mathcal{B}$  are distinct so the elements of  $\mathcal{B}$  are linearly independent. The result follows.  $\square$

Let  $\phi : \mathbb{Z}\langle X_5 \rangle \rightarrow \mathbb{Z}\langle X_5 \rangle / I$  be the natural epimorphism,  $\phi(f) = f + I$  for all  $f \in \mathbb{Z}\langle X_5 \rangle$ . Let  $W = W_1 + W_2$  and let  $U = \phi(W)$ ,  $U_i = \phi(W_i)$  ( $i = 1, 2$ ). Then  $\text{Ker } \phi \cap W = 0$  so  $\phi|_W : W \rightarrow U$  is an isomorphism. Hence, Lemmas 3.2 and 3.3 imply the following assertions.

**Corollary 3.4.**  $U_1 \cap U_2 = 0$ .

**Corollary 3.5.**  $U_2$  is a free abelian group with a basis  $\{b + I \mid b \in \mathcal{B}\}$ .

#### 4. PROOF OF PROPOSITION 2.5

It is clear that the additive group of the ring  $\mathbb{Z}\langle X_5 \rangle$  is a direct sum of the subgroups  $R_i$  ( $i \geq 0$ ) generated by the monomials of degree  $i$ ,

$$\mathbb{Z}\langle X_5 \rangle = \bigoplus_{i \geq 0} R_i.$$

It is also clear that  $T_5^{(4)}$  is generated as a subgroup of the additive group of  $\mathbb{Z}\langle X_5 \rangle$  by the polynomials  $a_1[a_2, a_3, a_4, a_5]a_6$  where  $a_i$  ( $1 \leq i \leq 6$ ) are monomials. It follows that  $T_5^{(4)} \cap R_i$  is generated as an additive group by the polynomials above such that  $\sum_{j=1}^6 \deg(a_j) = i$ .

Let  $\overline{R}_i = (R_i + I)/I$ ,  $\overline{T}_5^{(4)} = (T_5^{(4)} + I)/I$ . It is clear that  $\mathbb{Z}\langle X_5 \rangle / I = \overline{R}_0 \oplus \overline{R}_1 \oplus \dots \oplus \overline{R}_5$ ,  $\overline{T}_5^{(4)} = (\overline{R}_4 \cap \overline{T}_5^{(4)}) \oplus (\overline{R}_5 \cap \overline{T}_5^{(4)})$  and  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  is generated as a subgroup of the additive group of  $\mathbb{Z}\langle X_5 \rangle / I$  by the elements  $a_1[a_2, a_3, a_4, a_5]a_6 + I$  where  $a_i$  ( $1 \leq i \leq 6$ ) are monomials and  $\sum_{j=1}^6 \deg(a_j) = 5$ . It follows that  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  is generated as an additive group by

$$(4) \quad \begin{aligned} & x_{i_1}[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] + I, [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]x_{i_5} + I, [(x_{i_1}x_{i_2}), x_{i_3}, x_{i_4}, x_{i_5}] + I, \\ & [x_{i_1}, (x_{i_2}x_{i_3}), x_{i_4}, x_{i_5}] + I, [x_{i_1}, x_{i_2}, (x_{i_3}x_{i_4}), x_{i_5}] + I, [x_{i_1}, x_{i_2}, x_{i_3}, (x_{i_4}x_{i_5})] + I \end{aligned}$$

where  $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$ .

We claim that  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  is generated as an additive group by the elements

$$(5) \quad x_{i_1}[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] + I, [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] + I$$

and

$$(6) \quad \begin{aligned} & [x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}] + [x_{i_1}, x_{i_2}, x_{i_4}][x_{i_3}, x_{i_5}] + I, \\ & [x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}] + [x_{i_1}, x_{i_4}, x_{i_3}][x_{i_2}, x_{i_5}] + I. \end{aligned}$$

Indeed, it is straightforward to check repeating the calculations of the proof of Lemma 2.1 that all elements (4) belong to the additive group generated

by (5) and (6). On the other hand, it is clear that all elements (5) belong to  $\overline{R}_5 \cap \overline{T}_5^{(4)}$ . By Lemma 2.1, all elements (6) belong to  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  as well. Therefore, the elements (5) and (6) generate  $\overline{R}_5 \cap \overline{T}_5^{(4)}$ , as claimed.

Recall that  $U_i = \phi(W_i) = (W_i + I)/I$  ( $i = 1, 2$ ),  $U = U_1 + U_2$ . Since  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  is generated by the elements (5) and (6), we have  $\overline{R}_5 \cap \overline{T}_5^{(4)} \subseteq U$ .

Let  $U'_2$  be the subgroup of  $U_2$  generated by all elements  $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] + I$ . It follows easily from Lemma 3.1 that  $U'_2$  is a free abelian group and the set  $\{b + I \mid b \in \mathcal{B}_1\}$  is a basis of  $U'_2$ . Then, by Corollary 3.5,  $U/U'_2$  is a free abelian group with a basis formed by the images of the elements of  $\mathcal{B}_2$  and  $\mathcal{B}_3$ .

Note that  $U/(U_1 + U'_2) = (U_1 + U_2)/(U_1 + U'_2) \simeq U_2/(U_1 + U'_2) \cap U_2$ . By Corollary 3.4,  $(U_1 + U'_2) \cap U_2 = U'_2$  so  $U/(U_1 + U'_2) \simeq U_2/U'_2$ . Hence,  $U/(U_1 + U'_2)$  is a free abelian group with a basis formed by  $\overline{\mathcal{B}}_2 \cup \overline{\mathcal{B}}_3$  where  $\overline{\mathcal{B}}_j = \{b + (U_1 + U'_2) \mid b \in \mathcal{B}_j\}$  ( $j = 2, 3$ ).

Note also that  $U_1, U'_2 \subset \overline{R}_5 \cap \overline{T}_5^{(4)}$ . Since  $U_1 + U'_2$  is generated by the elements (5) and  $\overline{R}_5 \cap \overline{T}_5^{(4)}$  is generated by the elements (5) and (6), the quotient group  $(\overline{R}_5 \cap \overline{T}_5^{(4)})/(U_1 + U'_2)$  is generated by the images of elements (6).

Let  $H$  be the free (additive) abelian group freely generated by the set

$$\left\{ h_{i_1 i_2 i_3 i_4 i_5} \mid \{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\} \right\}.$$

Define a homomorphism  $\psi : H \rightarrow U/(U_1 + U'_2)$  by

$$\psi(h_{i_1 i_2 i_3 i_4 i_5}) = [x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}] + (U_1 + U'_2).$$

Let  $Q$  be the subgroup of  $H$  generated by the elements

$$h_{i_1 i_2 i_3 i_4 i_5} + h_{i_2 i_1 i_3 i_4 i_5}, h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_2 i_3 i_5 i_4}, h_{i_1 i_2 i_3 i_4 i_5} + h_{i_2 i_3 i_1 i_4 i_5} + h_{i_3 i_1 i_2 i_4 i_5}.$$

**Lemma 4.1.**  $\text{Ker } \psi = Q$ .

*Proof.* It can be easily seen that  $\psi(Q) = 0$  so  $Q \subseteq \text{Ker } \psi$ . Therefore, one can define a homomorphism  $\widehat{\psi} : H/Q \rightarrow U/(U_1 + U'_2)$  by  $\widehat{\psi}(h + Q) = \psi(h)$  for each  $h \in H$ . Let

$$\begin{aligned} \mathcal{C}_2 &= \left\{ h_{i_1 i_2 i_3 i_4 i_5} + Q \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, i_1 > i_2, i_3 \right\}, \\ \mathcal{C}_3 &= \left\{ h_{i_5 i_3 i_4 i_1 i_2} + Q \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, i_1 > i_2 \right\}. \end{aligned}$$

and let  $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$ . It can be easily checked that, modulo  $Q$ , each element  $h_{i_1 i_2 i_3 i_4 i_5}$  can be written as a linear combination of elements of  $\mathcal{C}$  so the quotient group  $H/Q$  is generated by the set  $\mathcal{C}$ . Note that  $\widehat{\psi}(\mathcal{C}_j) = \overline{\mathcal{B}}_j$  ( $j = 2, 3$ ); hence,  $\widehat{\psi}(\mathcal{C})$  is a basis of the free abelian group  $U/(U_1 + U'_2)$ . It follows that  $\text{Ker } \widehat{\psi} = 0$  so  $\text{Ker } \psi = Q$ , as required.  $\square$

Let  $P$  be the subgroup of  $H$  generated by  $Q$  together with all elements

$$h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_2 i_4 i_3 i_5}, h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_4 i_3 i_2 i_5}.$$

Recall that  $(\overline{R}_5 \cap \overline{T}_5^{(4)})/(U_1 + U'_2)$  is generated by the images of the elements (6), that is, by the images of  $[x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}] + [x_{i_1}, x_{i_2}, x_{i_4}][x_{i_3}, x_{i_5}] + I$  and  $[x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}] + [x_{i_1}, x_{i_4}, x_{i_3}][x_{i_2}, x_{i_5}] + I$  where  $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$ . These images coincide with  $\psi(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_2 i_4 i_3 i_5})$  and  $\psi(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_4 i_3 i_2 i_5})$ , respectively; since  $Q \subset P$ , we have  $P = \psi^{-1}((\overline{R}_5 \cap \overline{T}_5^{(4)})/(U_1 + U'_2))$ .

To complete the proof of Proposition 2.5 we need the following.

**Lemma 4.2.**  $h_{12345} \notin P$ .

*Proof.* Let  $\mu : H \rightarrow \mathbb{Z}$  be the homomorphism of  $H$  into  $\mathbb{Z}$  defined by

$$\mu(h_{i_1 i_2 i_3 i_4 i_5}) = \text{sgn}(\sigma)$$

where  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix}$ . Then

$$\begin{aligned} \mu(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_2 i_1 i_3 i_4 i_5}) &= \mu(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_2 i_3 i_5 i_4}) \\ &= \mu(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_2 i_4 i_3 i_5}) = \mu(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_1 i_4 i_3 i_2 i_5}) = 0 \end{aligned}$$

and

$$\mu(h_{i_1 i_2 i_3 i_4 i_5} + h_{i_2 i_3 i_1 i_4 i_5} + h_{i_3 i_1 i_2 i_4 i_5}) = \pm 3$$

so  $\mu(P) = 3\mathbb{Z}$ . On the other hand,  $\mu(h_{12345}) = 1 \notin 3\mathbb{Z}$  so  $h_{12345} \notin P$ .  $\square$

Now we are in a position to complete the proof of Proposition 2.5. Let

$$\eta : U/(U_1 + U'_2) \rightarrow U/(\overline{R}_5 \cap \overline{T}_5^{(4)})$$

be the natural epimorphism and let

$$\overline{\psi} = \psi \circ \eta : H \rightarrow U/(\overline{R}_5 \cap \overline{T}_5^{(4)}).$$

Then  $\text{Ker } \overline{\psi} = \psi^{-1}((\overline{R}_5 \cap \overline{T}_5^{(4)})/(U_1 + U'_2)) = P$ . It follows from Lemma 4.2 that  $\overline{\psi}(h_{12345}) \neq 0$ , that is,  $[x_1, x_2, x_3][x_4, x_5] + (\overline{R}_5 \cap \overline{T}_5^{(4)}) \neq (\overline{R}_5 \cap \overline{T}_5^{(4)})$ . Hence,  $v + I = [x_1, x_2, x_3][x_4, x_5] + I \notin (\overline{R}_5 \cap \overline{T}_5^{(4)})$ . Since  $v + I \in \overline{R}_5$ , we have  $v + I \notin \overline{T}_5^{(4)} = (T_5^{(4)} + I)/I$ , that is,  $v \notin T_5^{(4)} + I$ . The proof of Proposition 2.5 is completed.

## 5. PROOF OF THEOREM 1.3

Let  $\mathbb{Q}$  be the field of rationals and let  $\mathbb{Q}\langle X \rangle$  be the free unitary associative  $\mathbb{Q}$ -algebra on the free generating set  $X$ ,  $\mathbb{Z}\langle X \rangle \subset \mathbb{Q}\langle X \rangle$ . Let  $T_{\mathbb{Q}}^{(3,2)}$  be the ideal in  $\mathbb{Q}\langle X \rangle$  generated by all elements  $[a_1, a_2, a_3][a_4, a_5]$  and  $[a_1, a_2, a_3, a_4]$  where  $a_i \in \mathbb{Q}\langle X \rangle$ . It is clear that  $T^{(3,2)} \subseteq T_{\mathbb{Q}}^{(3,2)} \cap \mathbb{Z}\langle X \rangle$ .

The idea of the proof of Theorem 1.3 is similar to one used in [2] to prove that the additive group of  $\mathbb{Z}\langle X \rangle/T^{(3)}$  is free abelian. We will define a certain set  $\mathcal{D} \subset \mathbb{Z}\langle X \rangle$  and prove that, on one hand,  $\{d + T_{\mathbb{Q}}^{(3,2)} \mid d \in \mathcal{D}\}$



is a  $\mathbb{Q}$ -basis of  $\mathbb{Q}\langle X \rangle / T_{\mathbb{Q}}^{(3,2)}$  and, on the other hand,  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$  generates the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$ . Then  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$  is a linearly independent generating set of the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$  so this additive group is free abelian (with a basis  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$ ).

The following lemma is well-known (see [5, Theorem 4.3], [7, Lemma 1], [8, Lemma 3], [9, Theorem 1]).

**Lemma 5.1.** *For all  $a_1, \dots, a_6 \in \mathbb{Q}\langle X \rangle$ ,*

$$[a_1, a_2][a_3, a_4][a_5, a_6] + [a_1, a_3][a_2, a_4][a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)}.$$

*Proof.* We have

$$\begin{aligned} [a_1 a_4, a_2, a_3][a_5, a_6] &= [a_1[a_4, a_2] + [a_1, a_2]a_4, a_3][a_5, a_6] \\ &= (a_1[a_4, a_2, a_3] + [a_1, a_3][a_4, a_2] + [a_1, a_2][a_4, a_3] \\ &\quad + [a_1, a_2, a_3]a_4)[a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)}. \end{aligned}$$

Since  $[a_1 a_4, a_2, a_3][a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)}$ ,  $a_1[a_4, a_2, a_3][a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)}$  and

$$[a_1, a_2, a_3]a_4[a_5, a_6] = a_4[a_1, a_2, a_3][a_5, a_6] + [a_1, a_2, a_3, a_4][a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)},$$

we have

$$[a_1, a_2][a_3, a_4][a_5, a_6] + [a_1, a_3][a_2, a_4][a_5, a_6] \in T_{\mathbb{Q}}^{(3,2)},$$

as required.  $\square$

Since  $[b_1, b_2][b_3, b_4] \equiv [b_3, b_4][b_1, b_2] \pmod{T_{\mathbb{Q}}^{(3,2)}}$  for all  $b_1, b_2, b_3, b_4 \in \mathbb{Q}\langle X \rangle$ , Lemma 5.1 implies the following.

**Corollary 5.2.** *For all  $k \geq 3$ , all  $\sigma \in S_{2k}$  and all  $a_1, \dots, a_{2k} \in \mathbb{Q}\langle X \rangle$ , we have*

$$\begin{aligned} (7) \quad & [a_1, a_2][a_3, a_4] \dots [a_{2k-1}, a_{2k}] \\ & \equiv \operatorname{sgn}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}][a_{\sigma(3)}, a_{\sigma(4)}] \dots [a_{\sigma(2k-1)}, a_{\sigma(2k)}] \pmod{T_{\mathbb{Q}}^{(3,2)}}. \end{aligned}$$

Note that the argument above shows also that the congruence (7) holds in  $\mathbb{Z}\langle X \rangle$  modulo  $T^{(3,2)}$ .

Let  $C$  be the unitary subalgebra in  $\mathbb{Q}\langle X \rangle$  generated by all commutators  $[x_{i_1}, x_{i_2}, \dots, x_{i_k}]$  where  $k \geq 2$ ,  $i_s \in \mathbb{N}$  for all  $s$ ,  $x_i \in X$  for all  $i$ . Let

$$\begin{aligned} \mathcal{D}'_0 &= \{1\}, \quad \mathcal{D}'_1 = \{[x_{i_1}, x_{i_2}] \mid i_1 < i_2\}, \quad \mathcal{D}'_2 = \{[x_{i_1}, x_{i_2}, x_{i_3}] \mid i_1 < i_2, i_1 \leq i_3\}, \\ \mathcal{D}'_3 &= \{[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \mid i_1 < i_2, i_3 < i_4, i_1 \leq i_3; \text{ if } i_1 = i_3 \text{ then } i_2 \leq i_4\}, \\ \mathcal{D}'_4 &= \{[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \dots [x_{i_{2k-1}}, x_{i_{2k}}] \mid k \geq 3, i_1 < i_2 < \dots < i_{2k}\}. \end{aligned}$$

Let  $\mathcal{D}' = \mathcal{D}'_0 \cup \mathcal{D}'_1 \cup \mathcal{D}'_2 \cup \mathcal{D}'_3 \cup \mathcal{D}'_4$ .

In the proof of the next lemma we make use of a result proved in [7] and [9].

**Lemma 5.3.** *The set  $\{d' + T_{\mathbb{Q}}^{(3,2)} \mid d' \in \mathcal{D}'\}$  is a basis of the vector space  $(C + T_{\mathbb{Q}}^{(3,2)})/T_{\mathbb{Q}}^{(3,2)}$  over  $\mathbb{Q}$ .*

*Proof.* Note first that  $C$  is spanned by the set  $\mathcal{D}'$  modulo  $T_{\mathbb{Q}}^{(3,2)}$ .

Indeed, it is clear that  $C$  is spanned by 1 and the products  $c_1 c_2 \dots c_m$  ( $m \geq 1$ ) where each  $c_l$  is a commutator of length  $k(l) \geq 2$ ,  $c_l = [x_{i_{l1}}, x_{i_{l2}}, \dots, x_{i_{lk(l)}}]$ . Note that  $c_i c_j = c_j c_i \pmod{T_{\mathbb{Q}}^{(3,2)}}$  for all  $i, j$ . Further, if, for some  $l$ ,  $c_l$  is a commutator of length  $k(l) \geq 4$  then  $c_1 c_2 \dots c_m \in T_{\mathbb{Q}}^{(3,2)}$ . If, for some  $l$ ,  $c_l = [x_{i_{l1}}, x_{i_{l2}}, x_{i_{l3}}]$  is a commutator of length 3 and  $m > 1$  then again  $c_1 c_2 \dots c_m \in T_{\mathbb{Q}}^{(3,2)}$ . It follows that  $(C + T_{\mathbb{Q}}^{(3,2)})/T_{\mathbb{Q}}^{(3,2)}$  is spanned by 1, the commutators  $[x_{i_1}, x_{i_2}, x_{i_3}] + T_{\mathbb{Q}}^{(3,2)}$  ( $i_s \in \mathbb{N}$ ) and the products  $c_1 c_2 \dots c_m + T_{\mathbb{Q}}^{(3,2)}$  where  $m \geq 1$  and each  $c_l$  is a commutator of length 2.

It is clear that in a generator  $[x_{i_1}, x_{i_2}] + T_{\mathbb{Q}}^{(3,2)}$  of  $(C + T_{\mathbb{Q}}^{(3,2)})/T_{\mathbb{Q}}^{(3,2)}$  we can assume  $i_1 < i_2$ . Similarly, in a generator  $[x_{i_1}, x_{i_2}, x_{i_3}] + T_{\mathbb{Q}}^{(3,2)}$  we may assume  $i_1 < i_2$ ,  $i_1 \leq i_3$  and in a generator  $[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] + T_{\mathbb{Q}}^{(3,2)}$  we may assume that  $i_1 < i_2$ ,  $i_3 < i_4$  and  $i_1 \leq i_3$ ; if  $i_1 = i_3$  we may also assume  $i_2 \leq i_4$ . Finally, it follows immediately from (7) that in a generator

$$[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \dots [x_{i_{2k-1}}, x_{i_{2k}}] + T_{\mathbb{Q}}^{(3,2)} \quad (k \geq 3)$$

we can assume  $i_1 < i_2 < i_3 < \dots < i_{2k}$ . Thus, the set  $\{d' + T_{\mathbb{Q}}^{(3,2)} \mid d' \in \mathcal{D}'\}$  spans the vector space  $(C + T_{\mathbb{Q}}^{(3,2)})/T_{\mathbb{Q}}^{(3,2)}$  over  $\mathbb{Q}$ , as claimed.

Now to prove Lemma 5.3 it remains to check that the set  $\{d' + T_{\mathbb{Q}}^{(3,2)} \mid d' \in \mathcal{D}'\}$  is linearly independent in  $\mathbb{Q}\langle X \rangle / T_{\mathbb{Q}}^{(3,2)}$ .

Let  $R(m_1, m_2, \dots, m_k)$  ( $k \geq 0$ ,  $m_k > 0$  if  $k > 0$ ) be the linear span in  $\mathbb{Q}\langle X \rangle$  of all monomials  $x_{i_1} \dots x_{i_s}$  that contain  $m_1$  variables  $x_1$ ,  $m_2$  variables  $x_2$ ,  $\dots$ ,  $m_k$  variables  $x_k$  and do not contain any other variable. A polynomial  $f \in \mathbb{Q}\langle X \rangle$  is called *multilinear* if  $f \in R(m_1, \dots, m_k)$  where  $k > 0$  and  $m_i \in \{0, 1\}$  for all  $i$ . It follows from [7, Lemma 3] or from [9, Theorem 1] that the multilinear elements of  $\mathcal{D}'$  are linearly independent modulo  $T_{\mathbb{Q}}^{(3,2)}$ . Hence, it remains to prove that each non-multilinear element of  $\mathcal{D}'$  is not equal, modulo  $T_{\mathbb{Q}}^{(3,2)}$ , to a linear combination of other elements of  $\mathcal{D}'$ .

Note that  $\mathbb{Q}\langle X \rangle$  is a direct sum of the vector subspaces  $R(m_1, m_2, \dots, m_k)$  ( $k \geq 0$ ,  $m_k > 0$  if  $k > 0$ ) and  $\mathbb{Q}\langle X \rangle / T_{\mathbb{Q}}^{(3,2)}$  is a direct sum of the subspaces  $(R(m_1, m_2, \dots, m_k) + T_{\mathbb{Q}}^{(3,2)})/T_{\mathbb{Q}}^{(3,2)}$ . The non-multilinear elements of  $\mathcal{D}'$  are as follows:

- (1)  $[x_{i_1}, x_{i_2}, x_{i_3}] \in \mathcal{D}'_2$  where  $i_1 < i_2$  and either  $i_3 = i_1$  or  $i_3 = i_2$ ;
- (2)  $[x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \in \mathcal{D}'_3$  where  $i_1 < i_2$ ,  $i_3 < i_4$  and either  $i_1 = i_3$ ,  $i_2 \leq i_4$  or  $i_1 < i_3$ ,  $i_2 = i_3$  or  $i_1 < i_3$ ,  $i_2 = i_4$ .

Each non-multilinear element  $d'$  above is the only element of  $\mathcal{D}'$  that belongs to the corresponding term  $R(m_1, \dots, m_k)$ . Hence, to prove that this non-multilinear element  $d'$  is not a linear combination of other elements of  $\mathcal{D}'$  modulo  $T_{\mathbb{Q}}^{(3,2)}$  it suffices to check that  $d'$  is not equal to 0 modulo  $T_{\mathbb{Q}}^{(3,2)}$ , that is,  $d' \notin T_{\mathbb{Q}}^{(3,2)}$ .

If  $d' \in \mathcal{D}'_2$  then  $d' \notin T_{\mathbb{Q}}^{(3,2)}$  because the elements of  $\mathcal{D}'_2$  are of degree 3 and  $T_{\mathbb{Q}}^{(3,2)}$  does not contain non-zero polynomials of degree less than 4. Therefore, it remains to check that  $d' \notin T_{\mathbb{Q}}^{(3,2)}$  for the non-multilinear elements of  $\mathcal{D}'_3$ . Since the ideal  $T_{\mathbb{Q}}^{(3,2)}$  is invariant under permutations of the set  $X = \{x_1, x_2, \dots\}$  of variables, it suffices to check that  $[x_1, x_3][x_2, x_3] \notin T_{\mathbb{Q}}^{(3,2)}$  and  $[x_1, x_2]^2 \notin T_{\mathbb{Q}}^{(3,2)}$ .

Suppose, in order to get a contradiction, that  $[x_1, x_3][x_2, x_3] \in T_{\mathbb{Q}}^{(3,2)}$ . Then  $[x_1, x_3 + x_4][x_2, x_3 + x_4] \in T_{\mathbb{Q}}^{(3,2)}$  so

$$\begin{aligned} [x_1, x_3 + x_4][x_2, x_3 + x_4] - [x_1, x_3][x_2, x_3] - [x_1, x_4][x_2, x_4] \\ = [x_1, x_3][x_2, x_4] + [x_1, x_4][x_2, x_3] \in T_{\mathbb{Q}}^{(3,2)}. \end{aligned}$$

However,  $[x_1, x_3][x_2, x_4] + [x_1, x_4][x_2, x_3]$  is a sum of 2 multilinear elements of  $\mathcal{D}'$  and, as it was mentioned above, the multilinear elements of  $\mathcal{D}'$  are linearly independent modulo  $T_{\mathbb{Q}}^{(3,2)}$ . This contradiction shows that  $[x_1, x_3][x_2, x_3] \notin T_{\mathbb{Q}}^{(3,2)}$ . One can prove in a similar way that  $[x_1, x_2]^2 \notin T_{\mathbb{Q}}^{(3,2)}$  as well.

This completes the proof of Lemma 5.3.  $\square$

An ideal  $T$  in  $\mathbb{Q}\langle X \rangle$  is called a  $T$ -ideal if  $\phi(T) \subseteq T$  for all endomorphisms  $\phi$  of  $\mathbb{Q}\langle X \rangle$ . It is clear that  $T_{\mathbb{Q}}^{(3,2)}$  is a  $T$ -ideal in  $\mathbb{Q}\langle X \rangle$ . The following assertion is a particular case of a result due to Drensky [4], see also [3, Theorem 4.3.11].

**Theorem 5.4** ([4]). *Let  $T$  be a  $T$ -ideal in  $\mathbb{Q}\langle X \rangle$  and let  $\mathcal{E} \subset \mathbb{Q}\langle X \rangle$  be a set such that  $\{e + T \mid e \in \mathcal{E}\}$  is a basis of the vector space  $(C + T)/T$  over  $\mathbb{Q}$ . Then the set*

$$\{x_{i_1}x_{i_2}\dots x_{i_k}e \mid k \geq 0, i_1 \leq i_2 \leq \dots \leq i_k, e \in \mathcal{E}\}$$

*is a  $\mathbb{Q}$ -basis of  $\mathbb{Q}\langle X \rangle/T$ .*

Let

$$\mathcal{D} = \{x_{i_1}x_{i_2}\dots x_{i_k}d' \mid k \geq 0, i_1 \leq i_2 \leq \dots \leq i_k, d' \in \mathcal{D}'\}.$$

By Theorem 5.4, Lemma 5.3 implies the following.

**Corollary 5.5.** *The set  $\{d + T_{\mathbb{Q}}^{(3,2)} \mid d \in \mathcal{D}\}$  is a basis of  $\mathbb{Q}\langle X \rangle/T_{\mathbb{Q}}^{(3,2)}$  over  $\mathbb{Q}$ .*

Theorem 1.3 is an immediate corollary of the following assertion.

**Lemma 5.6.** *The additive group of the quotient ring  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$  is a free abelian group with a basis  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$ .*

*Proof.* It is straightforward to check that, modulo  $T^{(3,2)}$ , each element of  $\mathbb{Z}\langle X \rangle$  can be written as a linear combination of elements of  $\mathcal{D}$ . Hence, the set  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$  generates the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$ .

On the other hand, by Corollary 5.5, the elements of  $\mathcal{D}$  are linearly independent modulo  $T_{\mathbb{Q}}^{(3,2)} \cap \mathbb{Z}\langle X \rangle$ . Since  $T^{(3,2)} \subseteq T_{\mathbb{Q}}^{(3,2)} \cap \mathbb{Z}\langle X \rangle$ , the elements of  $\mathcal{D}$  are linearly independent modulo  $T^{(3,2)}$  as well (and, since they generate  $\mathbb{Z}\langle X \rangle$  modulo  $T^{(3,2)}$ , we have  $T^{(3,2)} = T_{\mathbb{Q}}^{(3,2)} \cap \mathbb{Z}\langle X \rangle$ ).

Thus, the set  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$  is a linearly independent generating set of the additive group of  $\mathbb{Z}\langle X \rangle / T^{(3,2)}$  so the latter group is free abelian with a basis  $\{d + T^{(3,2)} \mid d \in \mathcal{D}\}$ .  $\square$

The proof of Theorem 1.3 is completed.

## 6. PROOF OF PROPOSITION 1.4

Let

$$S = \{[x_{j_1}, x_{j_2}, x_{j_3}][x_{j_4}, x_{j_5}] \mid j_1 < j_2 < j_3 < j_4 < j_5\}$$

and let  $I^{(3,2)}$  be the ideal in  $\mathbb{Z}\langle X \rangle$  generated by  $S$ .

**Lemma 6.1.**  $T^{(3,2)} = T^{(4)} + I^{(3,2)}$ .

*Proof.* Since  $S \subset T^{(3,2)}$ , we have  $T^{(4)} + I^{(3,2)} \subset T^{(3,2)}$ . Hence, it remains to check that  $T^{(3,2)} \subset T^{(4)} + I^{(3,2)}$ . Since, modulo  $T^{(4)}$ , the ideal  $T^{(3,2)}$  is generated by the elements  $[a_1, a_2, a_3][a_4, a_5]$  ( $a_i \in \mathbb{Z}\langle X \rangle$ ), it suffices to check that  $[a_1, a_2, a_3][a_4, a_5] \in T^{(4)} + I^{(3,2)}$  for all  $a_i$ . Clearly, one can assume that  $a_i$  are monomials.

We use induction on degree of the polynomial  $f = [a_1, a_2, a_3][a_4, a_5]$ . If  $\sum_{i=1}^5 \deg a_i = 5$  then  $f = [x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}]$ . If  $i_1, \dots, i_5$  are all distinct then, by (2),  $f \equiv s \pmod{T^{(4)}}$  for some  $s \in S$  so  $f \in T^{(4)} + I^{(3,2)}$ . If, on the other hand,  $i_k = i_l$  for some  $k \neq l$  then, by (2),  $f \equiv -f \pmod{T^{(4)}}$ , that is,  $2f \in T^{(4)}$ . Since, by Lemma 2.2,  $3f \in T^{(4)}$ , we have  $f \in T^{(4)}$ .

Now suppose that  $\sum_{i=1}^5 \deg a_i = k > 5$ . Suppose that for all monomials  $b_1, \dots, b_5 \in \mathbb{Z}\langle X \rangle$  such that  $\sum_{i=1}^5 \deg b_i < k$  it has been already proved that  $[b_1, b_2, b_3][b_4, b_5] \in T^{(4)} + I^{(3,2)}$ . For some  $i$ ,  $1 \leq i \leq 5$ , we have  $a_i = a'_i a''_i$  where  $\deg a'_i, \deg a''_i < \deg a_i$ . By (2), we may assume without loss of generality that  $i = 5$ . Then

$$\begin{aligned} [a_1, a_2, a_3][a_4, a_5] &= [a_1, a_2, a_3][a_4, a'_5 a''_5] = [a_1, a_2, a_3]a'_5[a_4, a''_5] \\ &+ [a_1, a_2, a_3][a_4, a'_5]a''_5 \equiv a'_5[a_1, a_2, a_3][a_4, a''_5] + [a_1, a_2, a_3][a_4, a'_5]a''_5 \pmod{T^{(4)}}. \end{aligned}$$

By the inductive hypothesis,  $[a_1, a_2, a_3][a_4, a''_5], [a_1, a_2, a_3][a_4, a'_5] \in T^{(4)} + I^{(3,2)}$  so  $[a_1, a_2, a_3][a_4, a_5] \in T^{(4)} + I^{(3,2)}$ , as required.  $\square$

Let  $\xi : \mathbb{Z}\langle X \rangle \rightarrow \mathbb{Z}\langle X_4 \rangle$  be the projection such that  $\xi(x_i) = x_i$  if  $i \leq 4$  and  $\xi(x_i) = 0$  otherwise. Note that  $\xi(T^{(3,2)}) = T_4^{(3,2)}$ ,  $\xi(T^{(4)}) = T_4^{(4)}$ . Since  $\xi(S) = 0$ , we have  $\xi(I^{(3,2)}) = 0$  so, by Lemma 6.1,

$$T_4^{(3,2)} = \xi(T^{(3,2)}) = \xi(T^{(4)} + I^{(3,2)}) = \xi(T^{(4)}) + \xi(I^{(3,2)}) = \xi(T^{(4)}) = T_4^{(4)}.$$

It follows that  $T_m^{(3,2)} = T_m^{(4)}$  for all  $m$  such that  $2 \leq m \leq 4$ . The proof of Proposition 1.4 is completed.

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